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Existence and Uniqueness of Stationary Solutions for 3D Navier–Stokes System with Small Random Forcing via Stochastic Cascades

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We consider the 3D Navier–Stokes system in the Fourier space with regular forcing given by a stationary in time stochastic process satisfying a smallness condition. We explicitly construct a stationary solution of the system and prove a uniqueness theorem for this solution in the class of functions with Fourier transform majorized by a certain function *h*. Moreover we prove the following "one force—one solution" principle: the unique stationary solution at time *t* is presented as a functional of the realization of the forcing in the past up to *t*. Our explicit construction of the solution is based upon the stochastic cascade representation.

KEY WORDS: Navier–Stokes system; Stationary solution; Stochastic cascades; "One force–one solution" Principle.

1. INTRODUCTION: MAIN RESULT

The aim of this note is to prove an existence and uniqueness theorem for stationary solutions of randomly forced Navier–Stokes system on the 3D-torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ and on $\mathbb{R}^3$ with the help of the stochastic cascade representation of solutions introduced in (16) and developed in (1). Results on existence and uniqueness of stationary solutions to randomly and stochastically forced Navier–Stokes system in 2D can be found in(3,4,6–9,11–14,16). Existence-uniqueness theorems for stationary solutions of the Navier–Stokes system in 3D bounded domains are proved in (11) and (5). Our approach is completely different from that of (5,11). Using the techniques of (16) we construct an explicit representation of the stationary solution

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as a functional of the forcing realization in the past. There is a fixed point argument hidden behind our method, so a smallness assumption on the forcing is essential; it imposes a limitation on applicability of the approach. In particular, we prove existence and uniqueness of a stationary solution only in the class of functions majorized by a certain function h. However the solutions we construct belong to the Le Jan–Sznitman class and may have infinite energy, so that our new result is not covered by the results cited above.

In this section we introduce necessary notation and state an existence-uniqueness result for the Cauchy problem from [(1)] as well as our new result for stationary solutions. In Section 2 we describe our main tool, stochastic cascades, and prove the main result.

We consider the Cauchy problem for the Navier–Stokes system on $G = \mathbb{T}^3$ or $\mathbb{R}^3$:

$$\frac{\partial u(x,t)}{\partial t} + \langle u, \nabla\rangle u(x,t) = \nu\Delta u(x,t) - \nabla p(x,t) + g(x,t), \tag{1}$$

$$\langle \nabla, u\rangle = 0, \tag{2}$$

$$u(x,t_0) = u_0(x). \tag{3}$$

Here $x \in G$, and $u(x,t) \in \mathbb{R}^3$ is a divergence-free velocity field for each time $t \in [t_0,\infty)$. Angular brackets denote the Euclidean inner product, ∇ is the gradient operator, Δ is the Laplacian, $\nu > 0$ is the viscosity, $p : G \to \mathbb{R}$ is the pressure and $g : G \times [t_0,\infty) \to \mathbb{R}^3$ is the external forcing.

We shall assume that the initial data u_0 and the force g are divergence-free and zero mean:

$$\int_G u_0(x)dx = 0, \qquad \int_G g(x,t)dx = 0, \quad t \geqslant t_0.$$

Let us rewrite the Navier–Stokes system (1)—(3) in the Fourier space and get rid of the pressure term. Consider the case $G = \mathbb{T}^3$ first:

$$\begin{aligned}\frac{\partial \widehat{u}(k,t)}{\partial t} &= -4\pi^2\nu|k|^2\widehat{u}(k,t) - 2\pi i P_{k^\perp}\sum_{l_1+l_2=k}\langle k, \widehat{u}(l_1,t)\rangle\widehat{u}(l_2,t)\\ &\quad + \widehat{g}(k,t), \quad k \neq 0, \end{aligned} \tag{4}$$

$$\widehat{u}(0,t) = 0.$$

Here $u(x,t) = \sum_{k\in\mathbb{Z}^3}\widehat{u}(k,t)e^{2\pi i\langle k,x\rangle}$, $g(x,t) = \sum_{k\in\mathbb{Z}^3}\widehat{g}(k,t)e^{2\pi i\langle k,x\rangle}$, and $P_{k^\perp}$ is the orthogonal projection along the vector $e_k = \frac{k}{|k|}$ which corresponds to the projection on the space of divergence-free vector fields since the condition of zero divergence is expressed in the Fourier space as $\langle\widehat{u}(k,t), k\rangle = 0$ for all t and k.

We introduce a new function $\chi(k,t)$ via

$$\widehat{u}(k,t) = h(k)\chi(k,t), \tag{5}$$

where $h : \mathbb{Z}^3 \to \mathbb{R}_+$ is a normalizing function such that $h(k) > 0$ for $k \neq 0$ and $h(0) = 0$. The solutions we are going to construct will satisfy $|u(k,t)| \leqslant h(k)$. In particular, h will determine the behaviour of $u(k,t)$ at infinity. Further restrictions on h will be given in Theorems 1 and 2 and discussed in Section 2.

At this point we rewrite Eq. (4) as:

$$\begin{aligned}\chi(k,t) &= e^{-4\pi^2\nu(t-t_0)|k|^2}\chi(k,t_0)\\ &\quad+\frac{1}{2}\int_0^{t-t_0} 4\pi^2\nu|k|^2 e^{-4\pi^2\nu|k|^2 s} m(k) P_{k^\perp} \sum_{l_1+l_2=k} \langle e_k, \chi(l_1, t-s)\rangle\\ &\quad\chi(l_2, t-s) H(k, l_1, l_2) ds\\ &\quad+\frac{1}{2}\int_0^{t-t_0} 4\pi^2\nu|k|^2 e^{-4\pi^2\nu|k|^2 s}\varphi(k, t-s) ds, \quad k \neq 0. \end{aligned} \tag{6}$$

Here

$$m(k) = -\frac{4\pi i h * h(k)}{\nu|k| h(k)}, \quad H(k, l_1, l_2) = \frac{h(l_1)h(l_2)}{h * h(k)}, \quad \varphi(k,t) = \frac{2\widehat{g}(k,t)}{\nu|k|^2 h(k)} \tag{7}$$

where $h * h(k) = \sum_{l_1+l_2=k} h(l_1)h(l_2)$.

When $G = \mathbb{R}^3$ an analogous equation for the Fourier transform of u given by

$$\widehat{u}(k) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-i\langle k,x\rangle} u(x) dx$$

is

$$\begin{aligned}\chi(k,t) &= e^{-\nu(t-t_0)|k|^2}\chi(k,t_0)\\ &\quad+\frac{1}{2}\int_0^{t-t_0} \nu|k|^2 e^{-\nu|k|^2 s} m(k) P_{k^\perp}\\ &\quad\int_{\mathbb{R}^3}\langle e_k, \chi(l, t-s)\rangle \chi(k-l, t-s) H(k,l) dl ds\\ &\quad+\frac{1}{2}\int_0^{t-t_0} \nu|k|^2 e^{-\nu|k|^2 s}\varphi(k, t-s) ds, \quad k \neq 0, \end{aligned} \tag{8}$$

where now

$$m(k) = -\frac{2i h * h(k)}{\nu(2\pi)^{3/2}|k| h(k)}, \quad H(k,l) = \frac{h(l)h(k-l)}{h * h(k)}, \quad \varphi(k,t) = \frac{2\widehat{g}(k,t)}{\nu|k|^2 h(k)}$$

and $h * h(k) = \int_{\mathbb{R}^3} h(l)h(k-l) dl$ for a function $h : \mathbb{R}^3 \to \mathbb{R}_+$ such that $h(k) > 0$ for $k \neq 0$ and $h(0) = 0$. The function h will majorize the solution u thus determining its behaviour at zero and infinity. The function χ in (8) is a transformation of u defined by (5).

The following result was proved in [1] and [16]:

Theorem 1. *Let $G = \mathbb{T}^3$ (respectively, $\mathbb{R}^3$). Suppose $|m(k)| \leqslant 1$, $|\chi(k, t_0)| \leqslant 1$ for all k and $|\varphi(k, t)| \leqslant 1$ for all k and $t \in [t_0, t_1]$. Then there exists a solution of (6) (respectively, (8)) on $[t_0, t_1]$ with initial data $\chi(\cdot, t_0)$. This solution χ satisfies $|\chi(k, t)| \leqslant 1$ for all k and $t \in [t_0, t_1]$. It is unique in the space of functions bounded by 1 and defined on $\mathbb{Z}^3 \times [t_0, t_1]$ (respectively, $\mathbb{R}^3 \times [t_0, t_1]$).*

Corollary 1. *Suppose $|m(k)| \leqslant 1$, $|\widehat{u}(k, t_0)| \leqslant h(k)$, $|\widehat{g}(k, t)| \leqslant \nu|k|^2 h(k)/2$ for all k and t. Then there is a unique solution $\widehat{u}(k, t)$ of the Cauchy problem such that $|\widehat{u}(k, t)| \leqslant h(\cdot)$.*

Remark 1. *The condition on $|m(\cdot)|$ is fulfilled iff $4\pi h * h(k) \leqslant \nu|k|h(k)$ for $\mathbb{T}^3$ and $2h * h(k) \leqslant \nu(2\pi)^{3/2}|k|h(k)$ for $\mathbb{R}^3$. A possible choice of $h(\cdot)$ for both cases is*

$$h(k) = C_{\alpha,\beta} \frac{e^{-\alpha|k|^\beta}}{|k|^{2-\beta}},$$

with $\alpha > 0$, $0 \leqslant \beta \leqslant 1$ and sufficiently small $C_{\alpha,\beta}$. In particular, as $k \to \infty$ the solutions decay like $|k|^{-2}$ when $\beta = 0$ and like $e^{-\alpha|k|}$ when $\beta = 1$, see [1,16].

The main new result of this paper is concerned with the case of the external forcing given by a stochastic process. Now $\varphi(k, t) = \varphi(k, t, \omega)$ where ω is an element of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 2. *Let $G = \mathbb{T}^3$ (respectively, $\mathbb{R}^3$). Suppose $|m(k)| \leqslant 1$ for all $k \neq 0$, and the external forcing φ is a stationary process taking values in the space of functions defined on $\mathbb{Z}^3$ (respectively, $\mathbb{R}^3$) and bounded by 1: $|\varphi(k, t)| \leqslant 1$ for all k and t. Then there exist a solution of (6) (respectively, (8)) defined for $t \in \mathbb{R}$ which is a stationary process. This stationary solution χ satisfies $|\chi(k, t)| \leqslant 1$ for all k and t. It is the only solution of (6) defined for all $t \in \mathbb{R}$ with this property.*

The following "one force — one solution" principle holds: There exists a functional Ψ of forcing realizations on $\mathbb{R}_- = (-\infty, 0]$ such that the unique stationary solution χ is given by $\chi(\cdot, t) = \Psi(\pi_t\varphi)$ where $\pi_t\varphi$ defined by $\pi_t\varphi(\cdot, s) = \varphi(\cdot, t + s)$, $s \in \mathbb{R}_-$ is the history of the forcing φ up to time t shifted to $\mathbb{R}_-$.

Corollary 2. *Suppose $|m(k)| \leqslant 1$ for all $k \neq 0$. Let the forcing g be a stationary in time process with $|\widehat{g}(k, t)| \leqslant \nu|k|^2 h(k)/2$. Then there exists a unique stationary solution to the Navier–Stokes system satisfying $|\widehat{u}(k, t)| \leqslant h(k)$, $k \neq 0$.*

Remark 2. *In particular this result is applicable if the force is constant with respect to time. Thus, one obtains an existence-uniqueness theorem for steady state of the Navier–Stokes system.*

Since the construction of the functional Ψ will be based on the notion of stochastic cascade which played the central role in the proof of Theorem 1 in [1] and [16], we start the next section with a brief discussion of stochastic cascades.

2. STOCHASTIC CASCADES AND PROOF OF THEOREM 2

The proof will be given only for the case $G = \mathbb{T}^3$. The case $G = \mathbb{R}^3$ is treated in the same way.

The basic notion of this section is that of stochastic cascade. To introduce it we need some notation. Let

$$V = \bigcup_{j=0}^{\infty} \{1, 2\}^j = \{\theta, (1), (2), (1, 1), \ldots\}$$

be a complete binary tree rooted at θ, where $\{1, 2\}^0 = \{\theta\}$.

For $v = (v_1, v_2, \ldots, v_m) \in V$ and $1 \leqslant n \leqslant m$ denote $v|n = (v_1, v_2, \ldots, v_n)$ and$v|0 = \theta$. The concatenation $(v_1, \ldots, v_m, u_1, \ldots, u_n)$ of $v = (v_1, v_2, \ldots, v_m) \in V$ and $u = (u_1, v_2, \ldots, u_n) \in V$ is denoted by (v, u).

For a finite subtree W of V rooted at θ we define ∂W to be the set of all leaves of W, where a leaf is a vertex of W with no children in W.

Informally, the stochastic cascade we need can be described as a branching random walk. A single particle corresponding to the root θ of V is placed at a point $(k, t) \in \mathbb{Z}^3 \times \mathbb{R}$ and then the process takes place in the reverse time. The particle waits an exponentially distributed length of time S_θ with parameter $4\pi^2\nu|k|^2$ and then, at time $t - S_\theta$ an independent coin κ_θ is tossed and either with probability $1/2$ the event $\{\kappa_\theta = 0\}$ occurs and the particle dies, or with probability $1/2$ one has $\{\kappa_\theta = 1\}$ and the particle branches into two particles which are placed at $(l_1, t - S_\theta)$ and $(l_2, t - S_\theta)$ where the positions l_1 and l_2 are chosen according to the probability distribution $H(k, l_1, l_2)$ defined in (7). This procedure is repeated independently for these new particles which correspond to vertices (1) and (2) of a complete binary tree.

More precisely, for a given $k \in \mathbb{Z}^3_* = \mathbb{Z}^3 \setminus \{0\}$ we need a stochastic process $(k_v, \kappa_v, S_v)_{v \in V}$ indexed by the vertices of V (or, equivalently, a corresponding probability measure $\mathbb{P}$) with the following properties:

1. $\mathbb{P}\{k_v \in \mathbb{Z}^3_*, \kappa_v \in \{0, 1\}, S_v \in \mathbb{R}_+\} = 1$ for all $v \in V$.
2. Random variables k_θ, κ_θ and S_θ are independent with $\mathbb{P}\{k_\theta = k\} = 1$. $\mathbb{P}\{\kappa_\theta = i\} = \frac{1}{2}$ for $i = 0, 1$ and $\mathbb{P}\{S_\theta > s\} = e^{-4\pi^2\nu|k|^2 s}$ for $s \geqslant 0$.

3. Let the concatenation (v, b) of $v \in V$ and $b \in \{1, 2\}$ denote a child vertex of v. Suppose W is a subtree of V rooted at θ and $v \in \partial W$. Then,

$$\begin{aligned}\mathbb{P}\{k_{(v,1)} &= l_1, \kappa_{(v,1)} = i_1, S_{(v,1)} \geqslant s_1, k_{(v,2)} \\ &= l_2, \kappa_{(v,2)} = i_2, S_{(v,2)} \geqslant s_2 | \mathcal{F}_W\} \\ &= \frac{1}{4} H(k_v, l_1, l_2) e^{-4\pi^2 \nu |l_1|^2 s_1} e^{-4\pi^2 \nu |l_2|^2 s_2}, \\ & l_1, l_2 \in \mathbb{Z}^3_*, i_1, i_2 \in \{0, 1\}, s_1, s_2 \geqslant 0,\end{aligned}$$

where $\mathcal{F}_W = \sigma\{(k_w, \kappa_w, S_w)_{w \in W}\}$ (the factor $1/4$ is the product of two Bernoulli probabilities $1/2$).

The construction described above is called a *stochastic cascade* emitted from k.

Let us now introduce a deterministic functional $X(k, t, (k_v, \kappa_v, S_v)_{v \in V}, \chi_0(\cdot), \varphi(\cdot, \cdot), t_0)$ of a realization of the stochastic cascade $(k_v, \kappa_v, S_v)_{v \in V}$ emitted from $k_\theta = k$, initial data χ_0, and external forcing φ. Define for $v \in V$

$$X_v = \begin{cases} \chi_0(k_v), & T_v \leqslant t_0 \\ \varphi(k_v, T_v), & T_v > t_0, \kappa_v = 0 \\ m(k_v) P_{k_v^\perp} \langle e_{k_v}, X_{(v,1)} \rangle X_{(v,2)}, & T_v > t_0, \kappa_v = 1 \end{cases} \tag{9}$$

where $T_v = t - B_v$ for $v \in V$ and let

$$X(k, t, (k_v, \kappa_v, S_v)_{v \in V}, \chi_0(\cdot), \varphi(\cdot, \cdot), t_0) = X_\theta$$

.

Define $\tau(k, t, t_0)$ to be the maximal subtree of V rooted at θ such that $\kappa_v = 1$ and $t - A_v \geqslant t_0$ for all $v \in \tau(k, t, t_0) \setminus \partial\tau(k, t, t_0)$. In the particle language, $\tau(k, t, t_0)$ is the genealogical tree for the particles involved in the branching process described in the beginning of this section with an additional truncation, any particle being killed as soon as it reaches the initial time t_0.

Since the tree $\tau(k, t, t_0)$ can be viewed as a truncated representation of a critical branching process (any particle has zero or two children with equal probabilities, so the mean number of children is 1) it is finite with probability 1 and one can evaluate X_θ recursively starting with the leaves of $\tau(k, t, t_0)$. In fact, one can apply the first and second lines of definition (9) to the leaves of the tree and then apply the bottom line of the definition to each vertex that has children with already defined values of X.

The following result is the core of the proof of Theorem 1:

Lemma 1. *(see [1,16]) Under the conditions of Theorem 1,*

$$\chi(k, t) = \mathsf{E} X(k, t, (k_v, \kappa_v, S_v)_{v \in V}, \chi_0(\cdot), \varphi(\cdot, \cdot), t_0)$$

(the expectation is taken over realizations of the stochastic cascade) is a unique solution of (6) with $|\chi(k,t)| \leqslant 1$.

Proof of Theorem 2: For each realization of the random forcing φ, each $k \in \mathbb{Z}^3_*$ and $t \in \mathbb{R}$ we introduce a stochastic cascade $(k_v, \kappa_v, S_v)_{v \in V}$ emitted from k as described above and a functional Z analogous to (9):

$$Z_v = \begin{cases} \varphi(k_v, T_v), & \kappa_v = 0, \\ m(k_v) P_{k_v^\perp} \langle e_{k_v}, Z_{(v,1)} \rangle Z_{(v,2)}, & \kappa_v = 1, \end{cases} \tag{10}$$

and

$$Z(k, t, (k_v, \kappa_v, S_v)_{v \in V}, \varphi(\cdot, \cdot)) = Z_\theta.$$

Define $\tau(k)$ as the maximal subtree of V such that $\kappa_v = 1$ for all $v \in \tau(k) \setminus \partial\tau(k)$. This is the genealogical tree for all particles involved in the branching process. The tree is finite with probability 1 as a representation of a critical branching process. One can evaluate Z_θ recursively starting with the leaves of the tree $\tau(k)$. For the leaves one can apply the first line of definition (10), and for each vertex that has children with already defined values of Z one can apply the second line of the definition.

Since

$$|m(k_v) P_{k_v^\perp} \langle e_{k_v}, Z_{(v,1)} \rangle Z_{(v,2)}| \leqslant |m(k_v)||Z_{(v,1)}||Z_{(v,2)}|,$$

iterative application of definition (10) and conditions $|m(k)| \leqslant 1$, $|\varphi(k,t)| \leqslant 1$ implies that Z_θ is a product of finitely many factors each bounded by 1 in absolute value. Hence it is bounded by 1 itself and the expectation

$$\mathsf{E}_\varphi Z(k, t, (k_v, \kappa_v, S_v)_{v \in V}, \varphi(\cdot, \cdot))$$

with respect to the stochastic cascade with fixed realization of the forcing φ is well-defined. Let us denote this expectation by $\chi(k,t)$ and show that it is a solution of (6) on any time interval $[t_0, \infty)$.

It is sufficient to show that

$$\begin{aligned} \chi(k,t) = \frac{1}{2} \int_0^\infty & 4\pi^2 \nu |k|^2 e^{-4\pi^2 \nu |k|^2 s} m(k) P_{k^\perp} \\ & \sum_{l_1 + l_2 = k} \langle e_k, \chi(l_1, t-s) \rangle \chi(l_2, t-s) H(k, l_1, l_2) ds \\ & + \frac{1}{2} \int_0^\infty 4\pi^2 \nu |k|^2 e^{-4\pi^2 \nu |k|^2 s} \varphi(k, t-s) ds, \quad k \neq 0. \end{aligned} \tag{11}$$

To that end consider the following decomposition:

$$\mathsf{E}_\varphi Z_\theta = \mathsf{E}_\varphi Z_\theta \mathbf{1}\{\kappa_\theta = 1\} + \mathsf{E}_\varphi Z_\theta \mathbf{1}\{\kappa_\theta = 0\}. \tag{12}$$

Definition (10) implies

$$\mathsf{E}_{\varphi} Z_{\theta} \mathbf{1}\{\kappa_{\theta} = 0\} = \frac{1}{2} \int_0^{\infty} 4\pi^2 \nu |k|^2 e^{-4\pi^2 \nu |k|^2 s} \varphi(k, t - s) ds$$

so that the second term in (12) coincides with the one in (11). To treat the first term in (12) we need an easily verified lemma concerning stochastic cascades. It is the simplest form of the Markov property, though there are more complicated forms of the Markov property that also hold true. Let $K = (k_v, \kappa_v, S_v)_{v \in V}$ be a stochastic cascade emitted from k, and $u \in V$. Then we define the "shifted" stochastic cascade $K(u) = (\tilde{k}_v, \tilde{\kappa}_v, \tilde{S}_v)_{v \in V}$ via

$$(\tilde{k}_v, \tilde{\kappa}_v, \tilde{S}_v) = (k_{(u,v)}, \kappa_{(u,v)}, S_{(u,v)}), \quad v \in V.$$

Lemma 2. (Markov Property). *Stochastic cascades $K(1)$ and $K(2)$ are conditionally independent given $\kappa_\theta, S_\theta, k_1, k_2$ and the conditional distribution of $K(u)$ is a.s. equal to the distribution of a stochastic cascade emitted from k_u for $u = 1, 2$.*

We use this lemma to obtain

$$\begin{aligned}
\mathsf{E}_{\varphi} Z_{\theta} \mathbf{1}\{\kappa_{\theta} = 1\} &= \mathsf{E}_{\varphi}[m(k_\theta) P_{k_\theta^\perp} \langle e_{k_\theta}, Z_{(1)} \rangle Z_{(2)} \mathbf{1}\{\kappa_\theta = 1\}] \\
&= \mathsf{E}_{\varphi}[\mathsf{E}_{\varphi}[m(k_\theta) P_{k_\theta^\perp} \langle e_{k_\theta}, Z_{(1)} \rangle Z_{(2)} \mathbf{1}\{\kappa_\theta = 1\} | \kappa_\theta, S_\theta, k_1, k_2]] \\
&= m(k_\theta) \mathsf{E}_{\varphi}[\mathbf{1}\{\kappa_\theta = 1\} P_{k_\theta^\perp} \langle e_{k_\theta}, \mathsf{E}_{\varphi}[Z_{(1)} | \kappa_\theta, S_\theta, k_1, k_2] \rangle \\
&\qquad \times \mathsf{E}_{\varphi}[Z_{(2)} | \kappa_\theta, S_\theta, k_1, k_2]] \\
&= m(k_\theta) \mathsf{E}_{\varphi}[\mathbf{1}\{\kappa_\theta = 1\} P_{k_\theta^\perp} \langle e_{k_\theta}, \chi(k_1, t - S_\theta) \rangle \chi(k_2, t - S_\theta)] \\
&= m(k_\theta) \mathbb{P}\{\kappa_\theta = 1\} \mathsf{E}_{\varphi}[P_{k_\theta^\perp} \langle e_{k_\theta}, \chi(k_1, t - S_\theta) \rangle \chi(k_2, t - S_\theta)]
\end{aligned}$$

which is clearly equal to the first term in (11). Therefore, χ is a solution of (11) for all t.

Notice that functional Z_θ depends only on the random tree realization and the values of the forcing at the leaves of the tree. Since the random tree realization does not depend on t the resulting solution χ at time t is a functional of the realization of the forcing term φ in the past up to time t.

Suppose now that there is another solution $\gamma(k, t)$ of (6) which is bounded by 1 and defined for all $t \in \mathbb{R}$. Pick a $t_0 \in \mathbb{R}$ and consider $\gamma(k, t), t \geqslant t_0$ as the solution to the Cauchy problem for (6) with initial data $\gamma(k, t_0)$. Consider the stochastic cascade representations of χ and γ, the latter described in Lemma 1. Since $Z_\theta = X_\theta$ if $d(\tau(k)) < t - t_0$ where $d(\tau(k)) = \sup\{B_v : v \in \tau(k)\}$ is an a.s.-finite random variable, we obtain

$$|\chi(k, t) - \gamma(k, t)| \leqslant 2\mathbb{P}\{d(\tau(k)) \geqslant t - t_0\} \to 0 \quad \text{as } t_0 \to -\infty. \tag{13}$$

This implies that $\chi(k,t) = \gamma(k,t)$ and the theorem is proved. □

Remark 23. *An estimate similar to* (13) *shows that if one starts with arbitrary initial data* $\gamma(\cdot, t_0)$ *at some time* t_0 *then for fixed* k *the solution to the Cauchy problem* $\gamma(k,t)$ *approaches the stationary solution* $\chi(k,t)$ *as* $t \to \infty$. *But one cannot guarantee uniform convergence since the distribution of the lifetime* $d(\tau(k))$ *of a random branching process depends heavily on the initial particle position. The rate of decay in time of solutions in the absence of external forcing is discussed in* [2].

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REFERENCES

1. R. N. Bhattacharya, L. Chen, S. Dobson, R. B. Guenther, C. Orum, M. Ossiander, E. Thomann and E. C. Waymire, Majorizing kernels and stochastic cascades with applications to incompressible Navier-Stokes equations. *Trans. Am. Math. Soc.* **355**(12):5003–5040 (2003).
2. Y. Bakhtin, E. Dinaburg and Y. Sinai, On solutions of the Navier-Stokes system of infinite energy and enstrophy. In memory of A. A. Bolibrukh. *Uspekhi Mat. Nauk* **59**(6):55–72 (2004).
3. J. Bricmont, A. Kupiainen and R. Lefevere, Ergodicity of the 2D Navier-Stokes equations with random forcing. *Commun. Math. Phys.* **224**(1):65–81 (2001).
4. P.-L. Chow and R. Z. Khasminskii, Stationary solutions of nonlinear stochastic evolution equations. *Stochastic Anal. Appl.* **15**(5):671–699 (1997).
5. G. Da Prato and A. Debussche, Ergodicity for the 3D stochastic Navier-Stokes equations. *J. Math. Pures Appl.* **82**(8):877–947 (2003).
6. G. Da Prato and J. Zabczyk, Ergodicity for infinite dimensional systems. *London Mathematical Society Lecture Note Series*. 229 (Cambridge Univ. Press, Cambridge, 1996) xi, p. 339.
7. E. Weinan, J. C. Mattingly and Ya. Sinai, Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation. *Comm. Math. Phys.* **224**(1):83–106 (2001). Dedicated to Joel L. Lebowitz.
8. B. Ferrario, Ergodic results for stochastic Navier-Stokes equation. *Stochastics Stochastics Rep.* **60**(3–4):271–288 (1997).
9. F. Flandoli and B. Maslowski, Ergodicity of the 2-D Navier-Stokes equation under random perturbations. *Commun. Math. Phys.* **172**(1):119–141 (1995).
10. F. Flandoli and M. Romito, Statistically stationary solutions to the 3-D Navier-Stokes equation do not show singularities. *Electron. J. Probab.* **6**(5):15 (2001). electronic only.
11. M. Hairer and J. C. Mattingly, Ergodic properties of highly degenerate 2D stochastic Navier-Stokes equations. *Comptes Rendus Mathematique* **339**(12):879–882 (2004).
12. S. Kuksin, A. Piatniski and A. Shirikyan, A coupling approach to randomly forced nonlinear PDE's. II. *Commun. Math. Phys.* **230**(1):81–85, (2002).

13. S. Kuksin and A. Shirikyan, Stochastic dissipative PDE's and Gibbs measures. *Commun. Math. Phys.* **213**(2):291–330, (2000).
14. S. Kuksin and A. Shirikyan, Coupling approach to white-forced nonlinear PDEs. *J. Math. Pures Appl.* **81**:567–602 (2002).
15. Y. Le Jan and A. S. Sznitman, Stochastic cascades and 3-dimensional Navier-Stokes equations. *Prob. Theory and Rel. Fields* **109**(3):343–366 (1997).
16. J. C. Mattingly, On Recent Progress for the Stochastic Navier–Stokes Equations. *Journées "Equations aux Dérivées Partielles" (Forges-les-Eaux, 2003)*, XV, Summer 2003.